STEP Mark Schemes 2016

Mathematics

STEP 9465/9470/9475

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Introduction

These mark schemes are published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by the Examiners and shows the main valid approaches to each question. It is recognised that there may be other approaches and if a different approach was taken in the exam these were marked accordingly after discussion by the marking team. These adaptations are not recorded here.

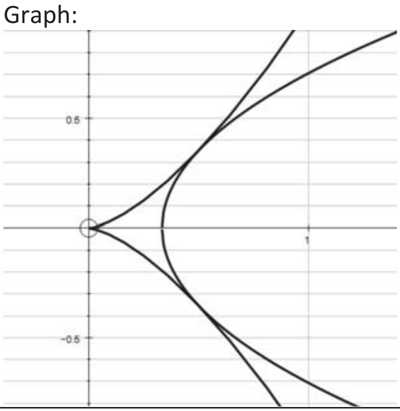
All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

The Admissions Testing Service will not enter into any discussion or correspondence in connection with this mark scheme.

STEP II 2016 MARK SCHEME

Question 1

	If the value of the parameter at P is p and the value of the parameter at Q is q:	
	Gradient of line OP is $\frac{p^3-0}{p^2-0}=p$ and similarly the gradient of OQ is q.	**M1**
	If the angle at O is a right angle, then $pq=-1$	**A1**
	$\frac{dx}{dt}=2t, \frac{dy}{dt}=3t^2$	**M1**
	Therefore $\frac{dy}{dx}=\frac{3}{2}t$	**A1**
	Equation of tangent at the point (t^2,t^3): $$y-t^3=\frac{3}{2}t(x-t^2)$$	**M1** **A1**
	$$\frac{3}{2}p(x-p^2)+p^3=\frac{3}{2}q(x-q^2)+q^3$$	**M1**
	$$3px-3p^3+2p^3=3qx-3q^3+2q^3$$	
	$$x=\frac{p^3-q^3}{3(p-q)}=\frac{1}{3}(p^2+pq+q^2)$$	**A1**
	Substitute for y: $$y-p^3=\frac{3}{2}p\left(\frac{1}{3}(p^2+pq+q^2)-p^2\right)+p^3$$	**M1**
	$$y=\frac{1}{2}pq(p+q)$$	**A1**
	Use $pq=-1$: $$x=\frac{p^4-p^2+1}{3p^2}$$	
	$$y=-\frac{p^2-1}{2p}$$	**B1**
	$$4y^2=\frac{p^4-2p^2+1}{p^2}=3x-1 \qquad (*)$$	**M1** **A1**
	If C_1 and C_2 meet then there must be a value of t such that: $$4t^6=3t^2-1$$	**B1**
	$$4t^6-3t^2+1=0$$	
	$$(2t^2-1)(2t^4+t^2-1)=0$$	**M1**
	$$(2t^2-1)^2(t^2+1)=0$$	**A1**
	Therefore, points of intersection only when $t=\pm\frac{\sqrt{2}}{2}$	**B1**
	Graph: 	**B1** **B1** **B1**

M1	An expression for the gradient of the line from the origin to a point on the curve. If applying Pythagoras to show that the angle is a right angle, the award M1 for a correct expression for the distance from the origin to a point on the curve.
A1	Correctly deducing that $pq = -1$
M1	Differentiation of both functions.
A1	Division to obtain correct gradient function.
M1	Attempt to find the equation of a tangent to the curve at one of the points.
A1	Correct equation.
M1	Elimination of one variable between the two tangent equations.
A1	Correct expression for either x or y found.
M1	Substitution to find the other variable.
A1	Correct expressions found for both variables.
B1	Using the relationship $pq = -1$ found earlier.
M1	An attempt to eliminate the parameter
A1	Fully correct reasoning leading to the equation given in the question.
B1	Condition for curves to meet identified.
M1	Attempt to factorise the equation.
A1	Correctly factorised.
B1	Points of intersection identified.
B1	Correct shape for $x = t^2, y = t^3$.
B1	Correct shape for $4y^2 = 3x - 1$.
B1	Graphs just touch at two points.

Question 2

	Let $c = a + b$: $$(2a + 2b)^3 - 6(2a + 2b)(a^2 + b^2 + (a + b)^2) + 8(a^3 + b^3 + (a + b)^3)$$	**M1**
	$$= 8(a + b)^3 - 24(a + b)(a^2 + ab + b^2) + 8(2a^3 + 3a^2b + 3ab^2 + 2b^3)$$	
	$$= 0$$	**M1**
	Therefore $(a + b - c)$ is a factor of (*)	**A1**
	By symmetry, $(b + c - a)$ and $(c + a - b)$ must also be factors.	**B1**
	So (*) must factorise to $k(a + b - c)(b + c - a)(c + a - b)$	**M1**
	To obtain the correct coefficient of a^3, $k = -3$	**M1**
	(*) factorises to $-3(a + b - c)(b + c - a)(c + a - b)$	**A1**
(i)	To match the equation given, we need $a + b + c = x + 1$, $a^2 + b^2 + c^2 = \frac{5}{2}$ and $a^3 + b^3 + c^3 = \frac{13}{4}$.	**M1**
	$$a = x, b = \frac{3}{2}, c = -\frac{1}{2}$$	**A1**
	The equation therefore factorises to $$-3(x + 2)(1 - x)(x - 2) = 0$$	**M1**
	$$x = -2, 1\ or\ 2$$	**A1**
(ii)	Let $d + e = c$ in (*): $a + b - d - e$ is a factor of $$(a + b + d + e)^2 - 6(a + b + d + e)(a^2 + b^2 + (d + e)^2) + 8(a^3 + b^3 + (d + e)^3)$$	
	Which is: $$(a + b + d + e)^2 - 6(a + b + d + e)(a^2 + b^2 + d^2 + e^2) + 8(a^3 + b^3 + d^3 + e^3)$$ $$-6(a + b + d + e)(2de) + 8(3d^2e + 3de^2)$$	**M1**
	$$-6(a + b + d + e)(2de) + 8(3d^2e + 3de^2) = -12ade - 12bde + 12d^2e + 12de^2$$	**M1**
	Which is $-12(a + b - d - e)(de)$. Therefore $(a + b - d - e)$ is a factor of: $$(a + b + d + e)^2 - 6(a + b + d + e)(a^2 + b^2 + d^2 + e^2) + 8(a^3 + b^3 + d^3 + e^3)$$	**A1**
	By symmetry, $a - b - d + e$ and $a - b + d - e$ must also be factors, so it must factorise to: $$k(a + b - d - e)(a - b - c + d)(a - b + c - d)$$	**M1**
	To obtain the correct coefficient we require $k = 3$.	**A1**
	To match the equation given we need $a + b + c + d = x + 6$, $a^2 + b^2 + c^2 + d^2 = x^2 + 14$ and $a^3 + b^3 + c^3 + d^3 = x^3 + 36$	**M1**
	$$a = x, b = 1, c = 2, d = 3$$	**A1**
	The equation therefore factorises to $$3x(x - 4)(x - 2)$$	**M1**
	$$x = 0, 2\ or\ 4$$	**A1**

M1	Substitution of $c = a + b$.
M1	Clear algebraic steps to show that the value of the function is 0.
A1	Conclusion that this means that $(a + b - c)$ is a factor.
B1	Identification of the other factors.
M1	Correct form of the factorisation stated.
M1	Consideration of any one coefficient to find the value of k.
A1	Correct factorisation.
M1	Identification of the equations that a, b and c must satisfy.
A1	Correct selection of a, b and c.
M1	Correct factorisation.
A1	Solutions of the equation.
M1	Substitution into the equation and rearrangement into the expression of the question and an extra term.
M1	Simplification of the extra term and factorisation.
A1	Conclusion.
M1	Identification of the other factors.
A1	Correct coefficient found.
M1	Identification of the equations that a, b, c and d must satisfy.
A1	Correct selection of a, b, c and d.
M1	Factorisation of the equation.
A1	Solutions found.

Question 3

(i)	$$f_n'(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$$	**B1**
(ii)	If a is a root of the equation then $f_n(a) = 0$	**B1**
	Each of the terms of $f(a)$ will be positive if $a > 0$.	**M1**
	Therefore $f_n(a) > 0$	**A1**
(iii)	$f_n'(a) = f_n(a) - \frac{a^n}{n!} = -\frac{a^n}{n!}$, and similarly for b.	**M1** **A1**
	Since a and b are both negative, $f_n'(a)$ and $f_n{}'(b)$ must have the same sign.	**M1** **M1**
	Therefore $f_n'(a)f_n'(b) > 0$	**A1**
	Two cases (positive and negative gradients) Sketch needed for each	**B1** **B1**
	Since the graph is continuous, there must be an additional root between a and b.	**M1** **A1**
	This would imply infinitely many roots.	**M1**
	But $f_n(x)$ is a polynomial of degree n, so has at most n roots	**M1**
	Therefore there is at most one root.	**A1**
	If n is odd then $f_n(x) \to -\infty$ as $x \to -\infty$ and $f_n(x) \to \infty$ as $x \to \infty$ There is one real root.	**M1** **A1**
	If n is even then $f_n(x) \to \infty$ as $x \to -\infty$ and $f_n(x) \to \infty$ as $x \to \infty$ There are no real roots.	**M1** **A1**

B1	Some explanation of the general term is required for this mark.
B1	Stated or implied elsewhere in the answer (such as when drawing conclusion).
M1	Clear statement about the individual terms.
A1	Clearly stated conclusion.
M1	Attempt to relate function to its derivative
A1	Correct relationship
M1	Statement that the signs must be the same.
M1	Consideration of the different cases for n.
A1	Conclusion that the product is positive.
B1	Sketch of graph with two roots with the curve passing through with positive gradient each time.
B1	Sketch of graph with two roots with the curve passing through with negative gradient each time. Second B1 may be given if only one graph sketched with a clear explanation of the similarities that the other graph would have.
M1	An attempt to explain that there must be a root between the two.
A1	Clear explanation including reference to continuity.
M1	Statement that this would imply infinitely many roots. OR Statement that the gradient would be negative or 0 at that root if the other two roots had positive gradients.
M1	Statement that there are at most n roots. OR Statement that a negative or zero gradient at the root in between would give a pair of roots contradicting the earlier conclusion.
A1	Conclusion.
M1	Correct identification of the outcome for n odd.
A1	A correct justification for the conclusion.
M1	Correct identification of the outcome for n even.
A1	A correct justification for the conclusion.

Question 4

(i)	$$y\cos\theta-\sin\theta=\frac{(x^2+x\sin\theta+1)\cos\theta-(x^2+x\cos\theta+1)\sin\theta}{x^2+x\cos\theta+1}$$ $$=\frac{(x^2+1)(\cos\theta-\sin\theta)}{x^2+x\cos\theta+1}$$	**M1** **A1**
	$$y-1=\frac{x(\sin\theta-\cos\theta)}{x^2+x\cos\theta+1}$$	**B1**
	$$(y\cos\theta-\sin\theta)^2=\frac{(x^2+1)^2(\sin\theta-\cos\theta)^2}{(x^2+x\cos\theta+1)^2}$$ $$=(y-1)^2\times\frac{(x^2+1)^2}{x^2}$$	**M1**
	$$\frac{(x^2+1)^2}{x^2}=\left(x+\frac{1}{x}\right)^2$$	**M1**
	Minimum value of $\left(x+\frac{1}{x}\right)^2$ is 4, therefore $(y\cos\theta-\sin\theta)^2\geq 4(y-1)^2$ $\quad(*)$	**M1** **A1**
	$y\cos\theta-\sin\theta$ can be written as $\sqrt{y^2+1}\cos(\theta+\alpha)$ for some value of α.	**M1** **A1**
	Therefore $y^2+1\geq(y\cos\theta-\sin\theta)^2\geq 4(y-1)^2$	**A1**
	$$y^2+1\geq 4y^2-8y+4$$	
	$$3y^2-8y+3\leq 0$$	**M1**
	Critical values are: $y=\frac{8\pm\sqrt{(8)^2-4(3)(3)}}{2(3)}$ $$\frac{4-\sqrt{7}}{3}\leq y\leq\frac{4+\sqrt{7}}{3}$$	**A1**
(ii)	If $y=\frac{4+\sqrt{7}}{3}$, then $\sqrt{y^2+1}=\sqrt{\frac{16+8\sqrt{7}+7}{9}+1}=\sqrt{\frac{32+8\sqrt{7}}{9}}$	
	$2(y-1)=\frac{2+2\sqrt{7}}{3}$	**M1**
	$\left(\frac{2+2\sqrt{7}}{3}\right)^2=\frac{4+8\sqrt{7}+28}{9}$, so $\sqrt{y^2+1}=2(y-1)$	**A1**
	Since $\sqrt{y^2+1}=2(y-1)$, the value of θ must be the value of α when $y\cos\theta-\sin\theta$ is written as $\sqrt{y^2+1}\cos(\theta+\alpha)$.	**B1**
	Therefore $\tan\theta=\frac{1}{y}=\frac{4-\sqrt{7}}{3}$	**M1** **A1**
	To find x: $$x^2y+xy\cos\theta+y=x^2+x\sin\theta+1$$	**M1**
	$$x^2(y-1)+x(y\cos\theta-\sin\theta)+y-1=0$$	
	Since $y\cos\theta-\sin\theta=\pm 2(y-1)$, and $y-1\neq 0$ this simplifies to: $$x^2\pm 2x+1=0$$	**M1**
	So we have either $x=1$ or $x=-1$	**A1**

M1	Substitution for y into $y\cos\theta - \sin\theta$.
A1	Correctly simplified.
B1	Correct simplification of $y - 1$.
M1	Relationship between $y\cos\theta - \sin\theta$ and $y - 1$.
M1	Simplification of the multiplier.
M1	Justification that the minimum value is 4.
A1	Conclusion that the given statement is correct.
M1	Calculation of the amplitude of $y\cos\theta - \sin\theta$.
A1	Correct value.
A1	Use to demonstrate the required result.
M1	Rearrangement to give quadratic inequality.
A1	Solve inequality and conclude the range given.
M1	Substitution of y into the two expressions.
A1	Demonstration that the equation holds.
B1	Statement that this must be an occasion where $y\cos\theta - \sin\theta$ takes its maximum value.
M1	Calculation of the value of $\tan\theta$.
A1	Correct simplification.
M1	Substitution to find x.
M1	Simplification of the equation to eliminate θ.
A1	Values of x found.

Question 5

(i)	Coefficient of x^n is $\frac{-N(-N-1)\dots(-N-n+1)}{n!}(-1)^n = \frac{N(N+1)\dots(N+n-1)}{n!}$ or $\binom{N+n-1}{N-1}$ or $\binom{N+n-1}{n}$	M1 M1 A1
	Expansion is therefore: $\sum_{r=0}^{\infty}\frac{N(N+1)\dots(N+r-1)}{r!}x^r$ or $\sum_{r=0}^{\infty}\binom{N+r-1}{N-1}x^r$	B1
	$(1-x)^{-1} = 1 + x + x^2 + \cdots$	B1
	Therefore the coefficient of x^n in the expansion of $(1-x)^{-1}(1-x)^{-N}$ is the sum of the coefficients of the terms up to x^n in the expansion of $(1-x)^{-N}$. $$\sum_{j=0}^{n}\binom{N+j-1}{j} = \binom{(N+1)+n-1}{n} = \binom{N+n}{n} \qquad (*)$$	M1 A1
(ii)	Write $(1+x)^{m+n}$ as $(1+x)^m(1+x)^n$.	B1
	When multiplying the two expansions, terms in x^r will be obtained by multiplying the term in x^j from one expansion by the term in x^{r-j} in the other expansion.	M1
	The coefficient of x^r in the expansion of $(1+x)^{m+n}$ is $\binom{m+n}{r}$	M1
	The coefficient of x^j in the expansion of $(1+x)^m$ is $\binom{m}{j}$	M1
	The coefficient of x^{r-j} in the expansion of $(1+x)^n$ is $\binom{n}{r-j}$	M1
	Therefore, summing over all possibilities: $$\binom{m+n}{r} = \sum_{j=0}^{n}\binom{m}{j}\binom{n}{r-j} \qquad (*)$$	A1
(iii)	Write $(1-x)^N$ as $(1-x)^{N+m}(1-x)^{-m}$	B1
	The coefficient of x^n in the expansion of $(1-x)^N$ is $(-1)^n\binom{N}{n}$	M1
	The coefficient of x^{n-j} in $(1-x)^{N+m}$ is $\binom{N+m}{n-j}(-1)^{n-j}$	M1 A1
	The coefficient of x^j in $(1-x)^{-m}$ is $\binom{m+j-1}{j}$	M1
	Therefore $$\sum_{j=0}^{n}\binom{N+m}{n-j}(-1)^{n-j}\binom{m+j-1}{j} = (-1)^n\binom{N}{n}$$	M1
	And so, $$\sum_{j=0}^{n}\binom{N+m}{n-j}(-1)^{j}\binom{m+j-1}{j} = \binom{N}{n} \qquad (*)$$	A1

M1	Full calculation written down.
M1	(-1) factors in all terms dealt with.
A1	Correct expression.
B1	Expansion written using summation notation.
B1	Expansion of $(1-x)^{-1}$.
M1	Sum that will make up the coefficient of x^n identified.
A1	Full explanation of given result.
B1	Correct splitting of the expression.
M1	Identification of the pairs that are to be multiplied together.
M1	Correct statement of the coefficient in the expansion of $(1+x)^{m+n}$
M1	Correct statement of the coefficient in the expansion of $(1+x)^{m}$
M1	Correct statement of the coefficient in the expansion of $(1+x)^{n}$
A1	Correct conclusion. Note that the answer is given, so each step must be explained clearly to receive the mark.
B1	Correct splitting of the expression.
M1	Correct statement of the coefficient in the expansion of $(1-x)^{N}$.
M1	Attempt to get the coefficient in the expansion of $(1-x)^{N+m}$ – award the mark if negative sign is incorrect.
A1	Correct coefficient.
M1	Correct statement of the coefficient in the expansion of $(1-x)^{-m}$.
M1	Combination of all of the above into the sum.
A1	Correct simplification.

Question 6

(i)	$$(1-x^2)\left(\frac{dy}{dx}\right)^2+y^2=1$$	
	If $y=x$, then $\frac{dy}{dx}=1$ and so LHS becomes $$(1-x^2)(1)^2+(x)^2=1=RHS$$	**B1**
	$y_1(1)=1$, so the boundary condition is also satisfied.	**B1**
(ii)	$$(1-x^2)\left(\frac{dy}{dx}\right)^2+4y^2=4$$	
	If $y=2x^2-1$, then $\frac{dy}{dx}=4x$ and so LHS becomes $$(1-x^2)(4x)^2+4(2x^2-1)^2=16x^2-16x^4+4(4x^4-4x^2+1)$$ $$=4=RHS$$	**M1** **A1**
	$y_2(1)=2(1)^2-1=1$, so the boundary condition is also satisfied.	**B1**
(iii)	If $z(x)=2\big(y_n(x)\big)^2-1$, then $\frac{dz}{dx}=4y_n(x)\frac{dy_n}{dx}$	**M1** **A1**
	Substituting in to the LHS of the differential equation: $$(1-x^2)\left(4y_n\frac{dy_n}{dx}\right)^2+4n^2(2(y_n)^2-1)^2$$	**M1**
	$$=16y_n^2\left[(1-x^2)\left(\frac{dy_n}{dx}\right)^2+n^2y_n^2-n^2\right]+4n^2$$	**M1** **A1**
	Since y_n is a solution of (*) when $k=n$: $$=4n^2$$	**A1**
	Since $z(1)=2(1)^2-1=1$, z is a solution to (*) when $k=2n$.	**M1**
	Therefore $y_{2n}(x)=2\big(y_n(x)\big)^2-1$	**A1**
(iv)	$$\frac{dv}{dx}=\frac{dy_n}{dx}\big(y_m(x)\big)\frac{dy_m}{dx}(x)$$	**B1**
	Substituting into LHS of (*) with $k=mn$: $$(1-x^2)\left(\frac{dy_n}{dx}\big(y_m(x)\big)\frac{dy_m}{dx}(x)\right)^2+(mn)^2\Big(y_n\big(y_m(x)\big)\Big)^2$$	**M1**
	$$=\frac{dy_n}{dx}\big(y_m(x)\big)\left((1-x^2)\left(\frac{dy_m}{dx}(x)\right)^2\right)+m^2n^2\Big(y_n\big(y_m(x)\big)\Big)^2$$	**M1**
	From (*), $(1-x^2)\left(\frac{dy_m}{dx}(x)\right)^2=m^2-m^2y_m(x)^2$	**M1**
	Therefore, we have: $$\frac{dy_n}{dx}\big(y_m(x)\big)(m^2-m^2y_m(x)^2)+m^2n^2\Big(y_n\big(y_m(x)\big)\Big)^2$$	
	Let $u=y_m(x)$, then this simplifies to $$m^2[(1-u^2)\frac{dy_n}{dx}(u)+n^2y_n(u)^2]$$	**M1**
	And by applying (*) when $k=n$, this simplifies to m^2n^2, so v satisfies (*) when $k=mn$.	**A1**
	$v(1)=y_n\big(y_m(1)\big)=y_n(1)=1$, so $v(x)=y_{mn}(x)$	**A1**

B1	Check that the function satisfies the differential equation.
B1	Check that the boundary conditions are satisfied.
M1	Differentiation and substitution.
A1	Confirm that the function satisfies the differential equation.
B1	Check that the boundary conditions are satisfied.
M1	Differentiation of z.
A1	Fully correct derivative.
M1	Substitution into LHS of the differential equation.
M1	Appropriate grouping.
A1	Expressed to show the relationship with the general differential equation.
A1	Use of the fact that y_n is a solution of the differential equation to simplify to the RHS.
M1	Check the boundary condition.
A1	Conclude the required relationship.
B1	Differentiation of v.
M1	Substitution into the correct version of the differential equation.
M1	Grouping of terms to apply the fact that y_m is a solution of a differential equation.
M1	Use of the differential equation.
M1	Simplification of the resulting expression.
A1	Conclusion that this simplified to m^2n^2
A1	Check of boundary condition and conclusion.

Question 7

	Let $y = a - x$:	
	Limits: $x = a \ \rightarrow y = 0$ $x = 0 \ \rightarrow y = a$	**B1**
	$$\frac{dy}{dx} = -1$$	**B1**
	$$\int_0^a f(x)\,dx = -\int_a^0 f(a-y)\,dy$$	
	Swapping limits of the integral changes the sign (and we can replace y by x in the integral on the right: $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$	**B1**
(i)	Using (*): $$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}\,dx = \int_0^{\frac{1}{2}\pi} \frac{\sin(\frac{1}{2}\pi - x)}{\cos(\frac{1}{2}\pi - x) + \sin(\frac{1}{2}\pi - x)}\,dx$$	**M1**
	$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}\,dx = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{\cos x + \sin x}\,dx$$	**A1**
	Therefore $$2\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}\,dx = \int_0^{\frac{1}{2}\pi} \frac{\sin x + \cos x}{\cos x + \sin x}\,dx = \int_0^{\frac{1}{2}\pi} 1dx = \frac{1}{2}\pi$$	**M1** **A1**
	$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}\,dx = \frac{1}{4}\pi$$	**A1**
(ii)	Using (*): $$\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x}\,dx = \int_0^{\frac{1}{4}\pi} \frac{\sin(\frac{1}{4}\pi - x)}{\cos(\frac{1}{4}\pi - x) + \sin(\frac{1}{4}\pi - x)}\,dx$$	
	$$\frac{\sin\left(\frac{1}{4}\pi - x\right)}{\cos\left(\frac{1}{4}\pi - x\right) + \sin\left(\frac{1}{4}\pi - x\right)} = \frac{\frac{\sqrt{2}}{2}(\cos x - \sin x)}{\frac{\sqrt{2}}{2}(\cos x + \sin x + \cos x - \sin x)} = \frac{1}{2}(1 - \tan x)$$	**M1** **M1** **A1**
	$$\frac{1}{2}\int_0^{\frac{1}{4}\pi} 1 - \tan x\,dx = \frac{1}{2}[x - \ln\lvert\sec x\rvert]_0^{\frac{1}{4}\pi}$$	**M1**
	$$= \frac{1}{8}\pi - \frac{1}{4}\ln 2$$	**A1**

(iii)	Using (*): $$\int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\,dx = \int_0^{\frac{1}{4}\pi} \ln\left(1+\tan\left(\frac{1}{4}\pi - x\right)\right)dx$$ $$= \int_0^{\frac{1}{4}\pi} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)\,dx$$ $$= \int_0^{\frac{1}{4}\pi} \ln\left(\frac{2}{1+\tan x}\right)\,dx$$	**M1**
	Therefore, if $I = \int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\,dx$, then $I = \frac{1}{4}\pi \ln 2 - I$	**M1**
	$$\int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\,dx = \frac{1}{8}\pi \ln 2$$	**A1**
(iv)	Using (*): $$I = \int_0^{\frac{1}{4}\pi} \frac{x}{\cos x\,(\cos x + \sin x)}\,dx = \int_0^{\frac{1}{4}\pi} \frac{\frac{1}{4}\pi - x}{\frac{\sqrt{2}}{2}(\cos x + \sin x)\sqrt{2}\cos x}\,dx$$ $$= \frac{1}{4}\pi \int_0^{\frac{1}{4}\pi} \frac{1}{(\cos x + \sin x)\cos x}\,dx - I$$	**M1**
	$$\int_0^{\frac{1}{4}\pi} \frac{1}{(\cos x + \sin x)\cos x}\,dx = \int_0^{\frac{1}{4}\pi} \frac{\sec^2 x}{1+\tan x}\,dx = [\ln(1+\tan x)]_0^{\frac{1}{4}\pi}$$	**M1** **A1**
	Therefore $$2I = \frac{1}{4}\pi \ln 2$$ $$I = \frac{1}{8}\pi \ln 2$$	**A1**

B1	Consideration of the limits of the integral.
B1	Completion of the substitution.
B1	Clear explanation about changing the sign when switching limits.
M1	Application of the given result.
A1	Simplification of the trigonometric ratios.
M1	Use of the relationship between the two integrals.
A1	Integration completed.
A1	Final answer.
M1	Correct replacement of either $\sin\left(\frac{1}{4}\pi - x\right)$ or $\cos\left(\frac{1}{4}\pi - x\right)$
M1	All functions of $\frac{1}{4}\pi - x$ replaced.
A1	Expression written in terms of $\tan x$.
M1	Integration completed.
A1	Limits substituted and integral simplified.
M1	Simplification of the integral.
M1	Use of properties of logarithms to reach an equation in I.
A1	Correct value.
M1	Rearrangement and split into two integrals.
M1	Rearrange to write in the form $\frac{f'(x)}{f(x)}$.
A1	Correct integration.
A1	Correct value for the original integral.

Question 8

	$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{m-\frac{1}{2}}^{\infty} = \frac{2}{2m-1}$$	**M1** **A1**
	Sketch of $y = \frac{1}{x^2}$	**B1**
	Rectangle drawn with height $\frac{1}{m^2}$ and width going from $m - \frac{1}{2}$ to $m + \frac{1}{2}$	**B1**
	Rectangle drawn with height $\frac{1}{n^2}$ and width going from $n - \frac{1}{2}$ to $n + \frac{1}{2}$	**B1**
	At least one other rectangle in between, showing that no gaps are left between the rectangles.	**B1**
	An explanation that the rectangle areas match the sum.	**B1**
(i)	Taking $m = 1$ and a very large value of n, the approximations for E is $2 - \frac{2}{2n+1}$	**M1**
	Therefore with $m = 1$, $E \to 2$ as $n \to \infty$	**A1**
	If $m = 2$, $E \to \frac{2}{3}$ as $n \to \infty$	**M1**
	Therefore an approximation for E is $1 + \int_{\frac{3}{2}}^{\infty} \frac{1}{x^2}dx = \frac{5}{3}$	**A1**
	Similarly, if $m = 3$, $E \to \frac{2}{5}$ as $n \to \infty$	
	Therefore an approximation for E is $1 + \frac{1}{4} + \int_{\frac{5}{2}}^{\infty} \frac{1}{x^2}dx = \frac{5}{4} + \frac{2}{5} = \frac{33}{20}$	**A1**
(ii)	$$\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{r-\frac{1}{2}}^{r+\frac{1}{2}} = \frac{2}{2r-1} - \frac{2}{2r+1} = \frac{4}{4r^2-1}$$	**M1** **A1**
	The error is $\frac{4}{4r^2-1} - \frac{1}{r^2} = \frac{1}{(4r^2-1)r^2} \approx \frac{1}{4r^4}$ for large values of r.	**M1** **A1**
	The error in the estimate for E is approximately $$\sum_{r=1}^{\infty} \frac{1}{r^4}$$	**B1**
	Using $E \approx \frac{33}{20}$, $$\sum_{r=3}^{\infty} \frac{1}{4r^4} \approx \frac{33}{20} - 1.645 = 0.005$$	**M1**
	Therefore: $$\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1 + 0.0625 + 4(0.005) = 1.083$$	**M1** **A1**

M1	Function integrated correctly.
A1	Limits applied.
B1	Sketch only required for positive x.
B1	Rectangle must have correct height.
B1	Rectangle must have correct height.
B1	It must be clear that there are no gaps between the rectangles.
B1	Clear justification.
M1	Correct expression for large n. Award this mark if the first integral from the question is used in the subsequent estimates.
A1	Correct explanation of the estimate in this case.
M1	Value of integral for the case $m = 2$.
A1	Add the first value.
A1	Apply the same process for $m = 3$.
M1	Evaluation of the integral with appropriate limits.
A1	Correct expression.
M1	Calculation of the error.
A1	Clear explanation that the given value is the correct approximation.
B1	Expression of the error as a sum.
M1	Use of most accurate estimate from part (i)
M1	Rearrangement to make the sum the subject.
A1	Correct answer.

Question 9

(i)	Kinetic energy lost by bullet is $\frac{1}{2}mu^2$	**M1**
	Work done against resistances is Ra	**M1**
	Energy lost = Work done	**M1**
	Therefore $a = \frac{mu^2}{2R}$.	**A1**
(ii)	Let v be the velocity of the combined block and bullet once the bullet has stopped moving relative to the block. Momentum is conserved, so $mu = (M+m)v$	**M1** **A1**
	In the case where the block was stationary, the bullet comes to rest over a distance of a, so its acceleration is $-\frac{u^2}{2a}$.	**M1** **A1**
	Consider the motion of the bullet until it comes to rest relative to the block: $$v^2 = u^2 + 2\left(-\frac{u^2}{2a}\right)(b+c)$$	**M1** **A1**
	Since $v = \frac{mu}{M+m}$: $$\left(\frac{mu}{M+m}\right)^2 = u^2 - \frac{u^2}{a}(b+c)$$	**M1**
	And so: $$a\left(\frac{m}{M+m}\right)^2 = a - b - c$$	**A1**
	The acceleration of the block must be $\frac{m}{M}$ times the acceleration of the bullet in the case where the block was fixed.	**M1**
	Therefore, the block accelerates from rest to a speed of $\frac{mu}{M+m}$ over a distance of c.	**M1**
	$'v^2 = u^2 + 2as'$: $$\left(\frac{mu}{M+m}\right)^2 = 0 + \frac{mu^2c}{Ma}$$	**M1** **A1**
	Therefore: $$\left(\frac{m}{M+m}\right)^2 = \frac{mc}{Ma}$$ and so $$c = \frac{mMa}{(M+m)^2}$$	**A1**
	Substituting to get b: $$a\left(\frac{m}{M+m}\right)^2 = a - b - \frac{mMa}{(M+m)^2}$$	**M1**
	$$b = a\left(1 - \frac{mM}{(M+m)^2} - \frac{m^2}{(M+m)^2}\right)$$	**M1**
	$$b = \frac{Ma}{(M+m)^2}$$	**A1**

M1	Calculation of the Kinetic Energy.
M1	Calculation of the work done.
M1	Statement that the two are equal.
A1	Rearrangement to give expression for a.
M1	Consideration of momentum.
A1	Correct relationship stated.
M1	Attempt to find the acceleration of the bullet.
A1	Correct expression found.
M1	Application of the acceleration found to the motion of the bullet when the block moves.
A1	Correct relationship found.
M1	Use of the relationship found from momentum considerations.
A1	Elimination of u from the equation.
M1	Statement of the relationship between the two accelerations.
M1	Correct identification of the other information relating to the uniform acceleration of the block.
M1	Use of $v^2 = u^2 + 2as$
A1	Relationship found.
A1	Simplification to get expression for c.
M1	Substitution into other equation.
M1	Rearrangement to make b the subject.
A1	Correct expression.

Question 10

	Find the centre of mass of the triangle: Let the two sides of the triangle with equal length have length b and the other side have length $2a$. Let $\bar{x}$ be the distance of the centre of mass from the side BC and along the line of symmetry.	**M1**
	$$(2a+2b)\bar{x} = 2b\left(\frac{1}{2}b\cos\theta\right)$$	**M1** **M1**
	$$\bar{x} = \frac{b^2\cos\theta}{2(a+b)}$$	**A1**
	Let the point of contact between the triangle and the peg be a distance y from the midpoint of BC. Let the weight of the triangle be W, the reaction force at the peg be R and the frictional force at the peg be F. Let the angle between BC and the horizontal be α.	**B1** **B1** **B1**
	Resolving parallel to BC: $$F = W\sin\alpha$$	**M1** **A1**
	Resolving perpendicular to BC: $$R = W\cos\alpha$$	**M1** **A1**
	$$\tan\alpha = \frac{y}{\bar{x}}$$	**B1**
	To prevent slipping: $$F \le \mu R$$	**M1**
	$$\mu \ge \tan\alpha$$	**A1**
	Therefore $$\mu \ge \frac{2y(a+b)}{b^2\cos\theta}$$	**M1** **A1**
	and y can take any value up to a, so the limit on μ is when $y = a$.	**M1**
	$$\mu \ge \frac{2a(a+b)}{b^2\cos\theta}$$	
	Since $a = b\sin\theta$:	**M1**
	$$\mu \ge \frac{2\sin\theta\,(\sin\theta+1)}{\cos\theta} = 2\tan\theta\,(1+\sin\theta)$$	**M1** **A1**

M1	Notations devised to allow calculations to be completed. May be seen on a diagram.
M1	Correct positions of centres of masses for individual pieces.
M1	Correct equation written down.
A1	Position of centre of mass found.
B1	Specification of a variable to represent the position of the centre of mass.
B1	Notations for all of the forces.
B1	An appropriate angle identified. (All three of these marks may be awarded for sight of the features on a diagram).
M1	Resolving in one direction.
A1	Correct equation stated. Must use angle θ.
M1	Resolving in another direction.
A1	Correct equation stated. Must use angle θ.
B1	Statement of the value of $\tan \alpha$.
M1	Use of coefficient of friction.
A1	Correct conclusion.
M1	Substitution for the angle.
A1	Correct inequality.
M1	Identification of the limiting case.
M1	Elimination of the side lengths.
M1	Inequality only in terms of θ found.
A1	Correct answer.

Question 11

(i)	Since the particles collide there is a value of t such that $$a + ut\cos\alpha = vt\cos\beta$$ $$ut\sin\alpha = b + vt\sin\beta$$	**M1**
	Multiply the first equation by b and make ab the subject: $$ab = bvt\cos\beta - but\cos\alpha$$	**M1**
	Multiply the second equation by a and make ab the subject: $$ab = aut\sin\alpha - avt\sin\beta$$	**M1**
	Equating: $$bvt\cos\beta - but\cos\alpha = aut\sin\alpha - avt\sin\beta$$ and so: $$aut\sin\alpha + but\cos\alpha = bvt\cos\beta + avt\sin\beta$$	**M1**
	$$aut\sin\alpha + but\cos\alpha = R_1\sin(\alpha + \theta_1)$$ where $R_1^2 = (aut)^2 + (but)^2$ and $\tan\theta_1 = \frac{b}{a}$	**M1** **A1** **A1**
	$$bvt\cos\beta + avt\sin\beta = R_2\sin(\beta + \theta_2)$$ where $R_2^2 = (avt)^2 + (bvt)^2$ and $\tan\theta_2 = \frac{b}{a}$	**M1** **A1** **A1**
	Since $\theta_1 = \theta_2$: $$R_1\sin(\theta + \alpha) = R_2\sin(\theta + \beta)$$ and since $vR_1 = uR_2$: $$u\sin(\theta + \alpha) = v\sin(\theta + \beta) \qquad (*)$$	**M1** **A1**
(ii)	Vertically: Bullet's height above the ground at time t is $b + vt\sin\beta - \frac{1}{2}gt^2$ Target's height above the ground at time t is $ut\sin\alpha - \frac{1}{2}gt^2$ Therefore the collision must occur when $t = \frac{b}{u\sin\alpha - v\sin\beta}$	**M1** **M1** **A1**
	The vertical height of the target at this time is $\frac{bu\sin\alpha}{u\sin\alpha - v\sin\beta} - \frac{1}{2}g\left(\frac{b}{u\sin\alpha - v\sin\beta}\right)^2$	**A1**
	If this is before it reaches the ground: $$\frac{bu\sin\alpha}{u\sin\alpha - v\sin\beta} - \frac{1}{2}g\left(\frac{b}{u\sin\alpha - v\sin\beta}\right)^2 > 0$$	**M1**
	Therefore: $$2bu\sin\alpha\,(u\sin\alpha - v\sin\beta) - b^2g > 0$$	
	$$2u\sin\alpha\,(u\sin\alpha - v\sin\beta) > bg$$	**A1**
	Both the bullet and target are affected equally by gravity, so any collision would correspond to the time for the straight line motion in part (i)	**B1**
	In part (i) there can clearly only be a collision if $\alpha > \beta$	**B1**

M1	Pair of equations stated.
M1	Make ab the subject of the first equation.
M1	Make ab the subject of the second equation.
M1	Put the two together.
M1	Rewrite in the form $R\sin(\alpha+\theta)$.
A1	Correct value of R.
A1	Correct value of $\tan\theta$.
M1	Rewrite in the form $R\sin(\beta+\theta)$.
A1	Correct value of R.
A1	Correct value of $\tan\theta$.
M1	Identify that the two values of θ are equal.
A1	Use the relationship between the values of R to reach the correct answer.
M1	Consider the motion of the bullet vertically.
M1	Consider the motion of the target vertically.
A1	Find the value of t for which the collision occurs.
A1	Substitute the value of t into one of the expressions for the height.
M1	State as an inequality.
A1	Rearrange to reach the required inequality.
B1	Relationship with part (i) identified.
B1	Required condition for a collision to take place in (i) identified.

Question 12

	$P(A \cup B \cup C) = P((A \cup B) \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$	**M1**
	$P((A \cup B) \cap C) = P((A \cap C) \cup (B \cap C))$ $= P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))$	**M1**
	$P((A \cap C) \cap (B \cap C)) = P(A \cap B \cap C)$	**M1**
	Therefore: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$	**A1**
	$P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D)$ $-P(A \cap B) - P(A \cap C) - P(A \cap D)$ $-P(B \cap C) - P(B \cap D) - P(C \cap D)$ $+P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D)$ $-P(A \cap B \cap C \cap D)$	**B1** **B1**
(i)	$P(E_i) = \frac{1}{n}$	**B1**
(ii)	There are a total of $n!$ arrangements possible.	**M1**
	$(n-2)!$ of these will have the ith and jth in the correct position.	**M1**
	$P(E_i \cap E_j) = \frac{1}{n(n-1)}$	**A1**
(iii)	By similar reasoning to (ii) the probability will be $\frac{1}{n(n-1)(n-2)}$	**M1**
		M1
		A1
	At least one card is in the position as the number it bears is the union of all of the E_is	**B1**
	$P\left(\bigcup_{1 \le i \le n} E_i\right) = \sum_{1 \le i \le n} P(E_i) - \sum_{1 \le i < j \le n} P(E_i \cap E_j) + \sum_{1 \le i < j < k \le n} P(E_i \cap E_j \cap E_k) - \cdots$ $+ (-1)^{n+1} P(E_1 \cap E_2 \cap \ldots \cap E_n)$	**M1**
	$P\left(\bigcup_{1 \le i \le n} E_i\right) = n \times \frac{1}{n} - \binom{n}{2} \times \frac{1}{n(n-1)} + \binom{n}{3} \times \frac{1}{n(n-1)(n-2)} - \cdots$ $+(-1)^{n+1} \times \frac{1}{n(n-1)(n-2)\ldots 2 \times 1}$	**M1** **M1**
	$P\left(\bigcup_{1 \le i \le n} E_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}$	**A1**
	The probability that no cards are in the same position as the number they bear is $\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$	**M1**
	Therefore the probability that exactly one card is in the same position as the number it bears is $n \times P(E_1) \times$ the probability that no card from a set of $(n-1)$ is in the same position as the number it bears.	
	$\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}$	**A1**

M1	Application of the given result applied for some splitting of $A \cup B \cup C$ into two sets.
M1	Correct handling of the intersection term in previous line.
M1	Intersections correctly interpreted.
A1	Fully correct statement.
B1	All pairwise intersections included.
B1	All other terms included.
B1	Correct answer.
M1	Total number of arrangements found. OR A tree diagram drawn.
M1	Number of arrangements where two are in the right place found. OR Correct probabilities on the tree diagram.
A1	Correct probability.
M1	Total number of arrangements found. OR A tree diagram drawn.
M1	Number of arrangements where two are in the right place found. OR Correct probabilities on the tree diagram.
A1	Correct probability.
B1	Identification of the required event in terms of the individual E_is.
M1	Use of the generalisation of the formula from the start of the question (precise notation not required).
M1	At least one of the individual sums worked out correctly.
M1	All of the parts of the sum worked out correctly.
A1	Correct answer.
M1	Probability of no card in correct position found.
A1	Correct answer.

Question 13

(i)	$X \sim B(16, \frac{1}{2})$ is approximated by $Y \sim N(8,4)$, so $P(X = 8) \approx P(\frac{15}{2} < Y < \frac{17}{2})$	**B1** **B1**
	In terms of $Z \sim N(0,1)$, this is $P(-\frac{1}{4} < Z < \frac{1}{4})$	**A1**
	The probability is therefore given by $$\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$$	**M1**
	This can be approximated as a rectangle with a width of $\frac{1}{2}$ and a height of $\frac{1}{\sqrt{2\pi}}$. The area is therefore $\frac{1}{2\sqrt{2\pi}}$ $$P(X = 8) \approx \frac{1}{2\sqrt{2\pi}}$$	**M1** **A1**
(ii)	$X \sim B(2n, \frac{1}{2})$ can be approximated by $Y \sim N(n, \frac{n}{2})$, so $P(X = n) \approx P(\frac{2n-1}{2} < Y < \frac{2n+1}{2})$	**B1** **B1**
	In the same way as part (i) $P(X = n)$ can be approximated by a rectangle of height $\frac{1}{\sqrt{2\pi}}$. The width will now be $\sqrt{\frac{2}{n}}$.	**M1** **A1**
	Therefore: $$P(X = n) = \frac{(2n)!}{n!\,n!}\left(\frac{1}{2}\right)^{2n} \approx \frac{1}{\sqrt{n\pi}}$$	**M1** **A1** **A1**
	Rearranging gives: $$(2n)! \approx \frac{2^{2n}(n!)^2}{\sqrt{n\pi}} \qquad (*)$$	**B1**
(iii)	$X \sim Po(n)$ can be approximated by $Y \sim N(n, n)$, so $P(X = n) \approx P(\frac{2n-1}{2} < Y < \frac{2n+1}{2})$	**B1**
	In the same way as part (i) $P(X = n)$ can be approximated by a rectangle of height $\frac{1}{\sqrt{2\pi}}$. The width will now be $\sqrt{\frac{1}{n}}$. The area is therefore $\frac{1}{\sqrt{2\pi n}}$.	**M1** **A1**
	Therefore:	
	$$\frac{e^{-n}n^n}{n!} \approx \frac{1}{\sqrt{2\pi n}}$$	**M1** **A1**
	Which simplifies to: $$n! \approx \sqrt{2\pi n}e^{-n}n^n$$	**A1**